UNIVERSITY EXAMINATION 2015/2016

SCHOOL OF PURE AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE

BED (SCIENCE), BED (ARTS) AND BSNE AND BAS
SCHOOL BASED

UNIT CODE: BMA3107   UNIT TITLE: REAL ANALYSIS

DATE: DECEMBER 2015   MAIN EXAM   TIME: 2 HOURS

Instructions; Answer ALL questions in Section A and any other TWO questions from Section B

QUESTION 1 (30 MARKS)

SECTION A

a). List the elements of the following sets:

i. \( A = \{ \text{prime numbers less than 20} \} \)  
ii. \( B = \{ \text{even numbers less than 16} \} \)  
iii. \( C = \{ \text{Multiplies of 4 less than 16} \} \)

Find:

iv. The intersection of the sets \( B \) and \( C \).  
v. The union of the sets \( A \) and \( B \).  
vi. The set difference \( B - A \).  
vii. If the universal set \( E = A \cup B \cup C \) find the complement of the set \( B \)

b). i. Define an Irrational number. Show that \( \sqrt{n} + 1 - \sqrt{n} - 1 \) is irrational for all \( n \in \mathbb{R}^+ \)

(5 Marks)
c). Define the concept of a countable set. Show that the set $Z$ of integers is countable. (5 Marks)

d). Determine the supremum and infimum of the following sets and state the one that belongs to the set.

i. \[ \left\{ \frac{1}{2}, \frac{3}{2}, \frac{4}{3}, \frac{4}{4}, ... \right\} \]

ii. \[ \left\{ -2, \frac{1}{2}, \frac{-3}{2}, \frac{1}{3}, \frac{-4}{3}, \frac{1}{4}, ... \right\} \] (4 Marks)

e). i). Define the concept of $\lim \text{sup} \ (x_n)$ and $\lim \text{inf} \ (x_n)$. (2 Marks)

ii). By computing $\lim \text{sup} \ (x_n)$ and $\lim \text{inf} \ (x_n)$ determine whether each of the following sequences diverges or converges in $\mathbb{R}$.

1. $x_n = (-1)^n (1 - \frac{1}{n})$ for all $n \in \mathbb{N}^+$ (2 Marks)

2. $x_n = \frac{n}{5} - \left\lfloor \frac{n}{5} \right\rfloor$ for all $n \in \mathbb{N}^+$ (2 Marks)

Where $[x]$ denotes the largest integer $\leq x$ (2 Marks)

**QUESTION 2 (20 MARKS)**

a) a). explain what is meant by the following:

(i) the set of natural numbers (2 Marks)
(ii) The set of integer (2 Marks)
(iii) A rational number (2 Marks)
(iv) The set of real numbers (2 Marks)

b) Give the difference between an even and an odd number (2 Marks)

c) Let $m$ be an integer. Show that $m$ is odd if and only if $m^2$ is odd (5 Marks)

d) Let $(S, <)$ be an ordered set and $E$ a subset of $S$, if the least upper bound of $E$ (lub. $E$) and the greatest lower bound of $E$ (glb. $E$) exist. Show that lub.$E$ is unique. (5 Marks)
QUESTION 3 (20 MQARKS)

a) Let \((X, d)\) be a metric space and \(A\) be a subset of \(X\) (Take \(X\) to be \(R\)) By use of examples define the following terms.

i) An open set \(A\) (2 Marks)

ii) A closed set \(A\) (2 Marks)

iii) Neighborhood of a point \(p \in X\) (2 Marks)

b) Prove that the set \(Q\) of irrational numbers is uncountable (4 Marks)

ii. Show that if \(t\) is irrational then \(s = \frac{t}{t-1}\) is also irrational. (4 Marks)

c) Show from first principles that the sequences \((x_n) = 1 + (-1)^n \frac{1}{n^2}\) converges to 1. (6 Marks)

QUESTION 4 (20 MARKS)

a) State the completeness axiom. Prove that if \(S\) is a bounded subset of real numbers then \(\text{Sup } S\) and \(\text{Inf } S\) both belong to \(S\) Iff \(S\) is closed (10 Marks)

b) Give the definition of the derivative of a function \(g(x)\). Prove that if \(g(x)\) is differentiable at \(C\) then \(g(x)\) is continuous at \(C\) (10 Marks)

QUESTION 5 (20 MARKS)

a) For all real numbers \(x, y\) show that:

i. \(|x+y| \leq |x| + |y|\) (4 Marks)

ii. \(|x-y| \geq ||x| - |y||\) (4 Marks)

b) Consider the sequence \((y_n)\) defined by \(y_1 = \sqrt{2}, y_{n+1} = \sqrt{2y_n}\) for all \(n \geq 2\). Use induction to show that the sequence is monotonic increasing and that \(y_n \leq 2\) for all \(n \in \mathbb{N}\). State with reasons whether \(y_n\) diverges or converges in \(R\). (7 Marks)

c) Show that for a bounded set \(S\) of real numbers there exists a number \(G > 0\) such that \(|x| \leq G \forall x \in S\) (5 Marks)